

Asymptotically free property of the solutions of an abstract linear hyperbolic equation with time-dependent coefficients

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Abstract

This paper is concerned with an abstract dissipative hyperbolic equation with time-dependent coefficient. Under an assumption which ensures that the energy does not decay, this paper provides a condition on the coefficient, which is necessary and sufficient so that the solutions tend to the solutions of the free wave equation.

Keywords: abstract linear hyperbolic equation; asymptotic behavior; asymptotically free property

1 Introduction

Let H be a separable complex Hilbert space H with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|$. Let A be a non-negative injective self-adjoint operator in H with domain $D(A)$. Let $c(t)$ be a function which is of bounded variation and satisfies

$$\inf_{t \geq 0} c(t) > 0. \quad (1.1)$$

We consider the initial value problem of the abstract dissipative wave equation

$$u''(t) + c(t)^2 Au(t) + b(t)u'(t) = 0 \quad t \geq 0, \quad (1.2)$$

$$u(0) = \phi_0, \quad u'(0) = \psi_0. \quad (1.3)$$

with time-dependent coefficients. There are a number of results concerning (1.2)–(1.3) (see, for example, [1, 8, 10, 11, 12, 6], [14, Section 2] and references therein).

In this paper, under the assumption that $b(t)$ is an integrable function on $[0, \infty)$, we give a necessary and sufficient condition for the existence of a wave speed c_* and a solution v of the free wave equation

$$v''(t) + c_*^2 Av(t) = 0, \quad (1.4)$$

satisfying

$$\lim_{t \rightarrow \infty} \left(\|A^{1/2}(u(t) - v(t))\| + \|u'(t) - v'(t)\| \right) = 0. \quad (1.5)$$

First, Arosio [1, Theorem 3] considered

$$\frac{\partial^2 u}{\partial t^2}(t, x) = a(t)\Delta u(t, x) + G(x, t) + H(x, t) \text{ in } [0, \infty) \times \Omega, \quad (1.6)$$

$$u(t, x) = 0 \quad \text{on } [0, \infty) \times \partial\Omega, \quad (1.7)$$

$$u(0, x) = \phi_0(x), \quad \frac{\partial u}{\partial t}(0, x) = \psi_0(x) \quad \text{in } \Omega. \quad (1.8)$$

for a bounded open set Ω in \mathbb{R}^n , where $a(t) = c(t)^2 + d(t)$ with $c(t)^2 \in BV(0, \infty)$ and $d(t) \in L^1(0, \infty)$ satisfying $0 < \nu \leq a(t)$ for almost every $t \in (0, \infty)$, $G \in L^1((0, \infty); L^2(\Omega))$, and $H \in BV((0, \infty); H^{-1}(\Omega))$ with $\lim_{t \rightarrow \infty} H(t) = 0$ in $H^{-1}(\Omega)$. Then he showed the following.

(i) If

$$\lim_{t \rightarrow \infty} \int_0^t (c(s) - c_\infty) ds \text{ exists and is finite,} \quad (1.9)$$

where

$$c_\infty = \lim_{t \rightarrow \infty} c(t),$$

then for every weak solution $u \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ of (1.6)–(1.7), there exists a solution $v \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ of the free wave equation

$$\frac{\partial^2 v}{\partial t^2}(t, x) = c_\infty^2 \Delta v(t, x) \text{ in } [0, \infty) \times \Omega, \quad (1.10)$$

$$v(t, x) = 0 \quad \text{on } [0, \infty) \times \partial\Omega, \quad (1.11)$$

satisfying

$$\lim_{t \rightarrow \infty} \left(\|u(t) - v(t)\|_{H_0^1(\Omega)} + \|u'(t) - v'(t)\|_{L^2(\Omega)} \right) = 0. \quad (1.12)$$

- (ii) Conversely, if there exists a weak solution $u(t) \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ of (1.6)–(1.7) and a non-trivial solution $v(t)$ of the free wave equation (1.10)–(1.11) such that (1.12) holds, then (1.9) must hold.

If we take $H = L^2(\Omega)$, $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $b(t) \equiv 0$, the abstract problem (1.2)–(1.3) becomes (1.6)–(1.7) above with $a(t) = c(t)^2$ and $G(t) \equiv H(t) \equiv 0$. The method of [1] is applicable for positive self-adjoint operators A with compact resolvent. Here we note that if $c(t)$ satisfies (1.1), then the assumptions $c^2(t) \in BV([0, \infty))$ and $c(t) \in BV([0, \infty))$ are equivalent.

Matsuyama [8, Theorem 2.1] considered the problem (1.6)–(1.7) for $\Omega = \mathbb{R}^n$, where $a(t) = c(t)^2$ with $c(t)$ satisfying (1.1) and

$$c \in Lip_{loc}([0, \infty)), \quad c' \in L^1(0, \infty), \quad (1.13)$$

$G(t) \equiv H(t) \equiv 0$, that is, the problem (1.2)–(1.3) with $H = L^2(\mathbb{R}^n)$, $A = -\Delta$ with $D(A) = H^2(\mathbb{R}^n)$ and $b(t) \equiv 0$, and showed the following: Assume that (1.9) holds. Then for every solution $u \in \bigcap_{j=0,1,2} C^j([0, \infty); H^{s-j}(\mathbb{R}^n))$ ($s \geq 1$) of (1.6)–(1.7), there exists a solution v of the free wave equation (1.10)–(1.11) satisfying

$$\lim_{t \rightarrow \infty} \left(\|\nabla(u(t) - v(t))\|_{H^{s-1}(\mathbb{R}^n)} + \|u'(t) - v'(t)\|_{H^{s-1}(\mathbb{R}^n)} \right) = 0. \quad (1.14)$$

On the other hand, he showed that if

$$\lim_{t \rightarrow \infty} \left| \int_0^t (c(s) - c_\infty) ds \right| = \infty, \quad (1.15)$$

there exists a non-trivial free solution u of (1.6)–(1.7) such that no solution v of the free wave equation (1.10)–(1.11) satisfies (1.14). Then, applying the result to Kirchhoff equation, he proved in [9] the existence of a non-trivial small initial data such that the solution of Kirchhoff equation is not asymptotically free.

Matsuyama and Ruzhansky [10, Theorem 1.1] considered the system $D_t U = A(t, D_x)U$ in $L^2(\mathbb{R}^n)^m$, and generalized the results of [8]. Furthermore, in a case $m = 1$ and $A(t, D_x) = -c(t)^2 \Delta$, this result is an improvement of the necessary condition for the asymptotically freeness of [8] as follows: Assume that c satisfies (1.1) and (1.13). If (1.15) holds, then for every non-trivial solutions of (1.6)–(1.7) with radially symmetric initial data, there exists no solution of the free wave equation (1.10)–(1.11) satisfying (1.14).

The purpose of this paper is to show a necessary and sufficient condition for asymptotically free property of (1.2)–(1.3) for general non-negative injective self-adjoint operator A (Theorem 1). Especially we are interested in the necessary condition. To obtain the necessary condition, Arosio [1, Theorem 3, (ii)] employed the discreteness of the spectrum corresponding to A , and Matsuyama and Ruzhansky [10, Theorem 1.1] employed the Riemann–Lebesgue theorem for the Fourier transform. In this paper, we use the property of continuous unitary group $e^{itA^{1/2}}$.

Another difference between the previous results and the result of this paper is that we do not assume $c_* = c_\infty$ in (1.4) a priori. We show that if there exists a non-trivial solution u of (1.2) which approaches to a solution of (1.4) with some wave speed c_* , then c_* coincides with $c_\infty = \lim_{t \rightarrow \infty} c(t)$ (Theorem 1 (ii)).

The result of this paper is applied to dissipative Kirchhoff equations in [15] to obtain the necessary decay condition on the dissipative term for the asymptotically free property. This condition is essentially stronger than that of linear dissipative wave equation.

2 Main result

Notation 1. For every $\alpha \geq 0$, the domain $D(A^\alpha)$ of A^α becomes a Hilbert space \mathbb{H}_α equipped with the inner product

$$(f, g)_{\mathbb{H}_\alpha} := (A^\alpha f, A^\alpha g)_H + (f, g)_H.$$

The norm is denoted by $\|f\|_{\mathbb{H}_\alpha}^2 = (f, f)_{\mathbb{H}_\alpha}$. We note that $\mathbb{H}_0 = H$. For every $\alpha < 0$, let \mathbb{H}_α denote the dual space of $\mathbb{H}_{-\alpha}$ with the dual norm, namely, \mathbb{H}_α is the completion of H by the norm

$$\|f\|_{\mathbb{H}_\alpha} = \sup\{|(f, g)_H|; g \in \mathbb{H}_{-\alpha}, \|g\|_{-\alpha} = 1\}.$$

Notation 2. For every $\alpha > 0$, let \mathcal{H}_α denote the completion of $D(A^\alpha)$ by the norm $\|A^\alpha \cdot\|$. Let \mathcal{A}^α be extension of A^α on \mathcal{H}_α . The fact that A^α is an injective self-adjoint operator implies that the range $R(A^\alpha)$ is dense in H , and thus $\mathcal{A}^\alpha : \mathcal{H}_\alpha \rightarrow H$ is bijective. From this fact and the definition, it follows that $\mathcal{A}^\alpha : \mathcal{H}_\alpha \rightarrow H$ is an isometric isomorphism.

Example 1. Let $H = L^2(\mathbb{R}^n)$ and $A = -\Delta$ with $D(A) = H^2(\mathbb{R}^n)$. For $\alpha > 0$, the space $\mathcal{H}_\alpha(\mathbb{R}^n)$ equals the homogeneous Sobolev space $\dot{H}^{2\alpha}$, and $\mathbb{H}_{-\alpha}$ equals the negative Sobolev space $H^{-2\alpha}(\mathbb{R}^n)$.

Notation 3. For a Banach space X , let $AC([0, \infty); X)$ denote all of X valued absolutely continuous functions on $[0, \infty)$, and $AC_{\text{loc}}([0, \infty); X) = \{f \in C([0, \infty)); f \in AC([0, T]) \text{ for every } T > 0\}$.

We consider the equation (1.2)–(1.3) and a free wave equation (1.4) in a somewhat wide class as

$$u''(t) + c(t)^2 A^{1/2} \mathcal{A}^{1/2} u(t) + b(t) u'(t) = 0, \quad t \geq 0, \quad (2.1)$$

$$u(0) = \phi_0, \quad u'(0) = \psi_0, \quad (2.2)$$

for $(\phi_0, \psi_0) \in \mathcal{H}_{1/2} \times H$, and

$$v''(t) + c_*^2 A^{1/2} \mathcal{A}^{1/2} v(t) = 0, \quad t \geq 0. \quad (2.3)$$

Definition 1. We say that u is a weak solution of (2.1) if $u \in C([0, \infty) : \mathcal{H}_{1/2})$,

$$\begin{aligned} u(t) - u(0) &\in \bigcap_{j=0,1} C^j([0, \infty); \mathbb{H}_{(1-j)/2}), \\ u'(t) &\in AC_{\text{loc}}([0, \infty); \mathbb{H}_{-1/2}), \end{aligned}$$

and (2.1) holds in the space $\mathbb{H}_{-1/2}$ for almost every $t \in (0, \infty)$.

A weak solution of (2.3) is defined as a weak solution of (2.1) with $c(t) \equiv c^*$ and $b(t) \equiv 0$.

Here we note that if u is a weak solution of (2.1)–(2.2), then $\mathbf{x} := (\mathcal{A}^{1/2} u, u')$ is a weak solution of the following Cauchy problem:

$$\frac{d}{dt} \mathbf{x}(t) + \begin{pmatrix} 0 & -A^{1/2} \\ c(t)^2 A^{1/2} & b(t) \end{pmatrix} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2.4)$$

$$\mathbf{x}(0) = \begin{pmatrix} \mathcal{A}^{1/2} \phi_0 \\ \psi_0 \end{pmatrix} \in H \times H, \quad (2.5)$$

in the sense that

$$\mathbf{x}(t) \in C([0, \infty); H \times H) \cap AC_{\text{loc}}([0, \infty); \mathbb{H}_{-1/2} \times \mathbb{H}_{-1/2}),$$

and that (2.4) holds in $\mathbb{H}_{-1/2} \times \mathbb{H}_{-1/2}$ for almost every $t \in (0, \infty)$. Conversely, if $\mathbf{x} = (w, z)$ is a weak solution of (2.4)–(2.5), then $u = \mathcal{A}^{-1/2} w$ is a weak solution of (2.1)–(2.2).

Our main result is the following:

Theorem 1. *Let $c(t)$ be of bounded variation on $(0, \infty)$ satisfying (1.1), and put $c_\infty = \lim_{t \rightarrow \infty} c(t)$. Let $b(t)$ be an integrable function on $[0, \infty)$. Then the following holds.*

- (i) *Suppose that (1.9) holds. Then for every weak solution u of (2.1), there exists a unique weak solution v of the free wave equation (2.3) with wave speed $c_* = c_\infty$ such that*

$$\lim_{t \rightarrow \infty} \left(\|\mathcal{A}^{1/2}(u(t) - v(t))\| + \|u'(t) - v'(t)\| \right) = 0 \quad (2.6)$$

holds.

- (ii) *Suppose that there exists a non-trivial weak solution u of (2.1), a positive constant c_* and a weak solution v of the free wave equation (2.3) such that (2.6) holds. Then $c_* = c_\infty$ and (1.9) must hold.*

Remark 1. *If $b(t)$ is integrable and of bounded variation as well, the Cauchy problem (2.1)–(2.2) is uniquely solvable. (See Proposition 5 in Appendix.)*

Remark 2. *Assume that the initial data $(\mathcal{A}^{1/2}\phi_0, \psi_0)$ belongs to $D(A^{J/2}) \times D(A^{J/2})$ for $J \geq 1$, and u is a solution of (2.1)–(2.2) in the sense that*

$$(\mathcal{A}^{1/2}u, u') \in C([0, \infty); \mathbb{H}_J \times \mathbb{H}_J) \cap AC_{\text{loc}}([0, \infty); \mathbb{H}_{J-1/2} \times \mathbb{H}_{J-1/2}), \quad (2.7)$$

and that (2.4) holds in $\mathbb{H}_{(J-1)/2} \times \mathbb{H}_{(J-1)/2}$ for almost every $t \in (0, \infty)$. Then the solution v of (2.3) given by (i) of Theorem 1 satisfies (2.7) and

$$\lim_{t \rightarrow \infty} \left(\left\| \mathcal{A}^{1/2}(u(t) - v(t)) \right\|_{\mathbb{H}_{J/2}} + \|u'(t) - v'(t)\|_{\mathbb{H}_{J/2}} \right) = 0. \quad (2.8)$$

In fact, since we see that (3.12) in section 2 with $\|\cdot\|_{H \times H}$ replaced by $\|\cdot\|_{\mathbb{H}_{J/2} \times \mathbb{H}_{J/2}}$ holds, we can prove (2.8) in the same way as in the proof of Theorem 1 (i).

3 Proof of Theorem 1

We first give a lemma, which is employed in the proof of the equality $c_* = c_\infty$.

Lemma 2. *If $g(t)$ is of bounded variation on $[0, \infty)$, then*

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T g(t) \exp(iG(t)A^{1/2})u \, dt \right\| = 0$$

for every $u \in H$, where $G(t) = \int_0^t g(s)ds$.

Proof. Let w be an arbitrary element of $D(A^{1/2})$. Then, $\exp(iG(t)A^{1/2})w$ is absolutely continuous on $[0, \infty)$ and differentiable almost everywhere on $(0, \infty)$, and thus we have

$$\frac{d}{dt} \exp(iG(t)A^{1/2})w = ig(t) \exp(iG(t)A^{1/2})A^{1/2}w. \quad (3.1)$$

for almost every t in $(0, \infty)$. Integrating (3.1) on $(0, T)$, and dividing the equality by T , we have

$$\frac{1}{T} \int_0^T \exp(iG(t)A^{1/2})g(t)A^{1/2}w \, dt = \frac{\exp(iG(T)A^{1/2})w - w}{iT}.$$

Since $\|\exp(i\tau(T)A^{1/2})w\| = \|w\|$ for every $T \in [0, \infty)$, we obtain

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T \exp(iG(t)A^{1/2})g(t)A^{1/2}w \, dt \right\| = 0.$$

Let $\delta > 0$ be an arbitrary positive number. The assumption that A is an injective self-adjoint operator implies that the range of $A^{1/2}$ is dense in H . Thus, we can take $w \in D(A^{1/2})$ such that $\|u - A^{1/2}w\| < \delta$, and therefore we have

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T g(t) \exp(iG(t)A^{1/2})u \, dt \right\| \\ & \leq \limsup_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T g(t) \exp(iG(t)A^{1/2})(u - A^{1/2}w) \, dt \right\| \\ & \quad + \lim_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T g(t) \exp(iG(t)A^{1/2})A^{1/2}w \, dt \right\| \\ & \leq \sup_{t \geq 0} \left(|g(t)| \left\| \exp(iG(t)A^{1/2})(u - A^{1/2}w) \right\| \right) \\ & \leq \delta \sup_{t \geq 0} |g(t)|. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we obtain

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{T} \int_0^T g(t) \exp(iG(t)A^{1/2})u \, dt \right\| = 0.$$

□

Now we prove Theorem 1. We express the solution $\mathbf{x}(t)$ of (2.4) by the method of ordinary differential equation by Wintner [13] (see also Coddington and Levinson [3], Hartman [7]), similarly to the proof of Matsuyama [8]. Let

$$Y(t) := \begin{pmatrix} e^{i\tau(t)A^{1/2}} & e^{-i\tau(t)A^{1/2}} \\ ic(t)e^{i\tau(t)A^{1/2}} & -ic(t)e^{-i\tau(t)A^{1/2}} \end{pmatrix},$$

where

$$\tau(t) = \int_0^t c(s)ds.$$

Then

$$Y(t)^{-1} = \frac{1}{2} \begin{pmatrix} e^{-i\tau(t)A^{1/2}} & -\frac{i}{c(t)}e^{-i\tau(t)A^{1/2}} \\ e^{i\tau(t)A^{1/2}} & \frac{i}{c(t)}e^{i\tau(t)A^{1/2}} \end{pmatrix}.$$

In order to approximate c by C^1 class functions, we use the mollifier as in the proof of Arosio [1]. Let ρ be a $C_0^\infty(\mathbb{R})$ function with support contained in $[-1, 1]$ and $\int_{\mathbb{R}} \rho(t)dt = 1$. Let δ be an arbitrary positive number. Put $\rho_\delta = \frac{1}{\delta}\rho(\frac{t}{\delta})$, and c_δ be the mollification of c , that is,

$$c_\delta(t) = \tilde{c} * \rho_\delta(t) = \int_{\mathbb{R}} \tilde{c}(t-s)\rho_\delta(s)ds (\in C^\infty(\mathbb{R})),$$

where \tilde{c} is a extension of c to \mathbb{R} such that $\tilde{c}(t) = c(0)$ for $t < 0$. From the assumption that c is bounded variation on $[0, \infty)$, it follows that

$$\int_S^T |c(s) - c_\delta(s)|ds \leq \delta \text{Var}(c; [\max\{S - \delta, 0\}, T + \delta]), \quad (3.2)$$

$$\int_S^T |c'_\delta(s)|ds \leq \text{Var}(c; [\max\{S - \delta, 0\}, T + \delta]), \quad (3.3)$$

for every $S, T \geq 0$ with $S < T$ (see [4] and [1]). Inequality (3.2) with $\delta = 1/n$ implies $\lim_{n \rightarrow \infty} c_{1/n} = c$ in $L^1((0, \infty))$. Thus, we can take a subsequence $\{n_k\}_{k=1}^\infty$ and a subset $N_1 \subset (0, \infty)$ such that the Lebesgue measure of N_1 is 0 and that

$$\lim_{k \rightarrow \infty} c_{1/n_k}(t) = c(t) \quad (3.4)$$

for every $t \in (0, \infty) \setminus N_1$. Let

$$Y_k(t) := \begin{pmatrix} e^{i\tau(t)A^{1/2}} & e^{-i\tau(t)A^{1/2}} \\ ic_{1/n_k}(t)e^{i\tau(t)A^{1/2}} & -ic_{1/n_k}(t)e^{-i\tau(t)A^{1/2}} \end{pmatrix}.$$

Then

$$Y_k(t)^{-1} = \frac{1}{2} \begin{pmatrix} e^{-i\tau(t)A^{1/2}} & -\frac{i}{c_{1/n_k}(t)} e^{-i\tau(t)A^{1/2}} \\ e^{i\tau(t)A^{1/2}} & \frac{i}{c_{1/n_k}(t)} e^{i\tau(t)A^{1/2}} \end{pmatrix},$$

and

$$\frac{d}{dt}Y_k(t) + \begin{pmatrix} 0 & -\frac{c(t)}{c_{1/n_k}(t)}A^{1/2} \\ c(t)c_{1/n_k}(t)A^{1/2} & -\frac{c'_{1/n_k}(t)}{c_{1/n_k}(t)^2} \end{pmatrix} Y_k(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From (1.1) and (3.4), it follows that

$$\lim_{k \rightarrow \infty} \|Y_k(t)^{-1} - Y(t)^{-1}\|_{\mathcal{L}(H \times H)} = 0 \quad \text{for every } t \in (0, \infty) \setminus N_1. \quad (3.5)$$

Let $\mathbf{x}(t)$ be a weak solution of (2.4)–(2.5). By putting

$$B_k(t) = Y_k(t)^{-1} \begin{pmatrix} 0 & \left(\frac{c(t)}{c_{1/n_k}(t)} - 1\right) \mathcal{A}^{1/2} \\ c(t)(c(t) - c_{1/n_k}(t)) \mathcal{A}^{1/2} & b(t) + \frac{c'_{1/n_k}(t)}{c_{1/n_k}(t)^2} \end{pmatrix} Y_k(t),$$

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} := Y(t)^{-1} \mathbf{x}(t),$$

and

$$\mathbf{y}_k(t) = \begin{pmatrix} y_k^{(1)}(t) \\ y_k^{(2)}(t) \end{pmatrix} := Y_k(t)^{-1} \mathbf{x}(t),$$

(2.4) is transformed into

$$\frac{d}{dt} \mathbf{y}_k(t) + B_k(t) \mathbf{y}_k(t) = 0 \quad \text{in } \mathbb{H}_{-1/2} \times \mathbb{H}_{-1/2}. \quad (3.6)$$

Let $\{E(\lambda)\}$ be a spectral family associated with the self adjoint operator A . Then (3.6) yields

$$\frac{d}{dt} (E(\lambda) \mathbf{y}_k)(t) + B_{\lambda,k}(t) (E(\lambda) \mathbf{y}_k)(t) = 0 \quad \text{in } H \times H, \quad (3.7)$$

for almost every $t \in (0, \infty)$, where

$$\begin{aligned} & B_{\lambda,k}(t) \\ &= Y_k(t)^{-1} \begin{pmatrix} 0 & \left(\frac{c(t)}{c_{1/n_k}(t)} - 1\right) \mathcal{A}^{1/2} E(\lambda) \\ c(t)(c(t) - c_{1/n_k}(t)) \mathcal{A}^{1/2} E(\lambda) & b(t) + \frac{c'_{1/n_k}(t)}{c_{1/n_k}(t)^2} \end{pmatrix} Y_k(t). \end{aligned}$$

By (1.1) and the fact that $e^{\pm isA^{1/2}}$ is unitary, the operators $Y_k(t)$ and $Y_k(t)^{-1}$ are bounded on $H \times H$ uniformly in k and t . Thus, observing (1.1) again, we have a positive constant K_1 satisfying

$$\|B_{\lambda,k}(t)\|_{\mathcal{L}(H \times H)} \leq K_1 \left(\lambda^{1/2} |c(t) - c_{1/n_k}(t)| + |c'_{1/n_k}(t)| + |b(t)| \right) \quad (3.8)$$

for every $\lambda, k > 0$ and every $t \geq 0$.

We estimate $(E(\lambda)\mathbf{y}_k)(t)$. The definition of weak solution implies $\mathbf{x}(t) \in AC_{\text{loc}}([0, \infty); \mathbb{H}_{-1/2} \times \mathbb{H}_{-1/2})$, and therefore, $(E(\lambda)\mathbf{y}_k)(t) \in AC_{\text{loc}}([0, \infty); H \times H)$. Thus, it follows from (3.7) and (3.8) that

$$\begin{aligned} & \|(E(\lambda)\mathbf{y}_k)(t) - (E(\lambda)\mathbf{y}_k)(s)\|_{H \times H} \\ & \leq K_1 \int_s^t \left(\lambda^{1/2} |c(\sigma) - c_{1/n_k}(\sigma)| + |c'_{1/n_k}(\sigma)| + |b(\sigma)| \right) \|(E(\lambda)\mathbf{y}_k)(\sigma)\|_{H \times H} d\sigma, \end{aligned} \quad (3.9)$$

for every $0 < s < t$. Thus

$$\begin{aligned} & \|(E(\lambda)\mathbf{y}_k)(t)\|_{H \times H} \leq \|(E(\lambda)\mathbf{y}_k)(0)\|_{H \times H} \\ & + K_1 \int_0^t \left(\lambda^{1/2} |c(\sigma) - c_{1/n_k}(\sigma)| + |c'_{1/n_k}(\sigma)| + |b(\sigma)| \right) \|(E(\lambda)\mathbf{y}_k)(\sigma)\|_{H \times H} d\sigma, \end{aligned}$$

for every $t \geq 0$. Hence by Gronwall's inequality together with the assumption that $b \in L^1((0, \infty))$, (3.2) and (3.3),

$$\begin{aligned} & \|(E(\lambda)\mathbf{y}_k)(t)\|_{H \times H} \\ & \leq \exp \left(K_1 \left((\lambda^{1/2}/n_k + 1) \text{Var}(c; [0, \infty)) + \|b\|_{L^1(0, \infty)} \right) \right) \|(E(\lambda)\mathbf{y}_k)(0)\|_{H \times H} \\ & \leq \exp \left(K_1 \left((\lambda^{1/2}/n_k + 1) \text{Var}(c; [0, \infty)) + \|b\|_{L^1(0, \infty)} \right) \right) \|\mathbf{y}_k(0)\|_{H \times H} \end{aligned}$$

for every $t \geq 0$. Substituting this inequality into (3.9), and observing (3.2) and (3.3) again, we obtain

$$\begin{aligned} & \|(E(\lambda)\mathbf{y}_k)(t) - (E(\lambda)\mathbf{y}_k)(s)\|_{H \times H} \\ & \leq K_1 \left((\lambda^{1/2}/n_k + 1) \text{Var}(c; [\max\{s - (1/n_k), 0\}, t + (1/n_k)]) + \|b\|_{L^1(s, t)} \right) \\ & \quad \times \exp \left(K_1 \left((\lambda^{1/2}/n_k + 1) \text{Var}(c; [0, \infty)) + \|b\|_{L^1(0, \infty)} \right) \right) \|\mathbf{y}_k(0)\|_{H \times H} \end{aligned} \quad (3.10)$$

for every $0 \leq s \leq t$. From (3.5), it follows that

$$\lim_{k \rightarrow \infty} \|\mathbf{y}_k(t) - \mathbf{y}(t)\|_{H \times H} = 0 \quad \text{for every } t \in (0, \infty) \setminus N_1,$$

and therefore

$$\lim_{k \rightarrow \infty} \|(E(\lambda)\mathbf{y}_k)(t) - (E(\lambda)\mathbf{y})(t)\|_{H \times H} = 0$$

for every $s, t \in (0, \infty) \setminus N_1$ and $\lambda > 0$. Thus, letting $k \rightarrow \infty$ in (3.10), we obtain

$$\begin{aligned} \|(E(\lambda)\mathbf{y})(t) - (E(\lambda)\mathbf{y})(s)\|_{H \times H} &\leq K_1 \left(\text{Var}(c; [s-, t+]) + \|b\|_{L^1(s, t)} \right) \\ &\quad \times \exp \left(K_1 \text{Var}(c; [0, \infty)) + \|b\|_{L^1(0, \infty)} \right) \|\mathbf{y}(0)\|_{H \times H} \end{aligned}$$

for every $s, t \in (0, \infty) \setminus N_1$ and $\lambda > 0$, where $\text{Var}(c; [s-, t+]) = \lim_{\delta \rightarrow 0+0} \text{Var}(c; [s - \delta, t + \delta])$. Therefore we have

$$\begin{aligned} \|\mathbf{y}(t) - \mathbf{y}(s)\|_{H \times H} &\leq K_1 \left(\text{Var}(c; [s-, t+]) + \|b\|_{L^1(s, t)} \right) \\ &\quad \times \exp \left(K_1 \text{Var}(c; [0, \infty)) + \|b\|_{L^1(0, \infty)} \right) \|\mathbf{y}(0)\|_{H \times H} \end{aligned} \quad (3.11)$$

for every $s, t \in (0, \infty) \setminus N_1$. Since c is of bounded variation on $[0, \infty)$, $\lim_{s, t \rightarrow \infty} \text{Var}(c; [s-, t+]) = 0$. Hence, letting $s, t (\notin N_1) \rightarrow \infty$ in (3.11) implies the existence of the limit

$$\lim_{t \notin N_1, t \rightarrow \infty} \mathbf{y}(t) = \mathbf{y}_\infty := \begin{pmatrix} y_{1, \infty} \\ y_{2, \infty} \end{pmatrix} \text{ in } H \times H.$$

Thus $\mathbf{y}(t)$ is expressed as

$$\mathbf{y}(t) = \mathbf{y}_\infty + \mathbf{r}(t),$$

with

$$\lim_{t \notin N_1, t \rightarrow \infty} \|\mathbf{r}(t)\|_{H \times H} = 0. \quad (3.12)$$

Hence we obtain the expression of the solution of (1.2)

$$\begin{pmatrix} \mathcal{A}^{1/2} u(t) \\ u'(t) \end{pmatrix} = \mathbf{x}(t) = Y(t)\mathbf{y}(t) = Y(t)\mathbf{y}_\infty + Y(t)\mathbf{r}(t) \quad (3.13)$$

$$= \begin{pmatrix} e^{i\tau(t)A^{1/2}} y_{1, \infty} + e^{-i\tau(t)A^{1/2}} y_{2, \infty} \\ ic(t) e^{i\tau(t)A^{1/2}} y_{1, \infty} - ic(t) e^{-i\tau(t)A^{1/2}} y_{2, \infty} \end{pmatrix} + Y(t)\mathbf{r}(t). \quad (3.14)$$

Let v be a solution of (1.4). Then it is expressed as

$$\mathbf{z}(t) = \begin{pmatrix} \mathcal{A}^{1/2} v(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} e^{ic_* t A^{1/2}} \phi + e^{-ic_* t A^{1/2}} \psi \\ ic_* e^{ic_* t A^{1/2}} \phi - ic_* e^{-ic_* t A^{1/2}} \psi \end{pmatrix}, \quad (3.15)$$

where

$$\phi = \frac{1}{2} \left(\mathcal{A}^{1/2} v(0) - \frac{i}{c_*} v'(0) \right), \quad \psi = \frac{1}{2} \left(\mathcal{A}^{1/2} v(0) + \frac{i}{c_*} v'(0) \right).$$

Since $\mathbf{x}, \mathbf{z} \in C([0, \infty); H \times H)$, we easily see that $\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{z}(t)\|_{H \times H} = 0$ if and only if

$$\lim_{t \notin N_1, t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{z}(t)\|_{H \times H} = 0. \quad (3.16)$$

Thus, the convergence (2.6) holds if and only if (3.16) holds. By the expressions (3.14) and (3.15), we see that (3.16) holds if and only if the following two convergences hold.

$$\lim_{t \notin N_1, t \rightarrow \infty} \|e^{i\tau(t)A^{1/2}} y_{1,\infty} + e^{-i\tau(t)A^{1/2}} y_{2,\infty} - e^{ic_* t A^{1/2}} \phi - e^{-ic_* t A^{1/2}} \psi\| = 0, \quad (3.17)$$

$$\begin{aligned} \lim_{t \notin N_1, t \rightarrow \infty} \|c(t)e^{i\tau(t)A^{1/2}} y_{1,\infty} - c(t)e^{-i\tau(t)A^{1/2}} y_{2,\infty} \\ - c_* e^{ic_* t A^{1/2}} \phi + c_* e^{-ic_* t A^{1/2}} \psi\| = 0. \end{aligned} \quad (3.18)$$

Here we prove the following lemma.

Lemma 3. *Assume that v is a weak solution of linear wave equation of (1.4) with*

$$c_* = c_\infty (= \lim_{t \rightarrow \infty} c(t)). \quad (3.19)$$

Then the convergence (2.6) holds if and only if the following two convergences hold:

$$\lim_{t \notin N_1, t \rightarrow \infty} \|e^{i(\tau(t) - c_\infty t)A^{1/2}} y_{1,\infty} - \phi\| = 0, \quad (3.20)$$

$$\lim_{t \notin N_1, t \rightarrow \infty} \|e^{-i(\tau(t) - c_\infty t)A^{1/2}} y_{2,\infty} - \psi\| = 0. \quad (3.21)$$

Proof. By the argument above, the convergence (2.6) holds if and only if (3.17) and (3.18) hold. By the assumption (3.19) and the fact that $e^{i\tau(t)A^{1/2}}$ is a C^0 unitary group on H , we see that (3.18) holds if and only if the following convergence holds.

$$\lim_{t \notin N_1, t \rightarrow \infty} \|e^{i\tau(t)A^{1/2}} y_{1,\infty} - e^{-i\tau(t)A^{1/2}} y_{2,\infty} - e^{ic_\infty t A^{1/2}} \phi + e^{-ic_\infty t A^{1/2}} \psi\| = 0. \quad (3.22)$$

Hence, (2.6) holds, if and only if (3.17) and (3.22) hold, equivalently, the following two convergences hold.

$$\begin{aligned}\lim_{t \notin N_1, t \rightarrow \infty} \|e^{i\tau(t)A^{1/2}} y_{1,\infty} - e^{ic_\infty t A^{1/2}} \phi\| &= 0, \\ \lim_{t \notin N_1, t \rightarrow \infty} \|e^{-i\tau(t)A^{1/2}} y_{2,\infty} - e^{-ic_\infty t A^{1/2}} \psi\| &= 0.\end{aligned}$$

Since $e^{isA^{1/2}}$ is a unitary group on H , these convergences are equivalent to (3.20) and (3.21). \square

Now we are ready to complete the proof of Theorem 1.

Proof of (i). Assume that (1.9) holds. We take $c_* = c_\infty (= \lim_{t \rightarrow \infty} c(t))$, and

$$\phi = e^{i \lim_{t \rightarrow \infty} (\tau(t) - c_\infty t) A^{1/2}} y_{1,\infty}, \quad \psi = e^{-i \lim_{t \rightarrow \infty} (\tau(t) - c_\infty t) A^{1/2}} y_{2,\infty}.$$

Then by the strong continuity of the $e^{isA^{1/2}}$ with respect to s on $[0, \infty)$, the convergences (3.20) and (3.21) hold, and therefore (2.6) holds by Lemma 3.

Proof of (ii). Assume that there are a non-trivial solution u of (2.1), a positive number c_* and a solution v of (2.3) such that (2.6) holds. Put

$$F(t) := \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} c(t)^2 \|\mathcal{A}^{1/2} u(t)\|^2$$

for every $t \geq 0$. Since u is non-trivial and $\|u'(t)\|^2 + \|\mathcal{A}^{1/2} u(t)\|^2$ is continuous, there is $S \in [0, \infty) \setminus N_1$ such that $\|u'(S)\|^2 + \|\mathcal{A}^{1/2} u(S)\|^2 > 0$. Then by (1.1), we have

$$F(S) > 0. \tag{3.23}$$

For every $\lambda > 0$ and $\delta > 0$, we put $u_\lambda = E([0, \lambda))u$,

$$\begin{aligned}F_\lambda(t) &:= \frac{1}{2} \|u'_\lambda(t)\|^2 + \frac{1}{2} c(t)^2 \|\mathcal{A}^{1/2} u_\lambda(t)\|^2 \quad \text{for every } t \geq 0, \\ F_{\lambda,k}(t) &:= \frac{1}{2} \|u'_\lambda(t)\|^2 + \frac{1}{2} c_{1/n_k}(t)^2 \|\mathcal{A}^{1/2} u_\lambda(t)\|^2 \quad \text{for every } t \geq 0.\end{aligned}$$

Since u satisfies (2.1) in $\mathbb{H}_{-1/2} \times \mathbb{H}_{-1/2}$ for almost every $t \in (0, \infty)$, u_λ

satisfies (2.1) in $H \times H$ for almost every $t \in (0, \infty)$. Thus we have

$$\begin{aligned}
F'_{\lambda,k}(t) &= (c_{1/n_k}(t)^2 - c(t)^2)(u'_\lambda(t), A^{1/2}\mathcal{A}^{1/2}u_\lambda(t))_H - b(t)\|u'_\lambda(t)\|^2 \\
&\quad + c_{1/n_k}(t)c'_{1/n_k}(t)\|\mathcal{A}^{1/2}u_\lambda(t)\|^2 \\
&\geq -\frac{1}{c_0}|c_{1/n_k}(t)^2 - c(t)^2|\sqrt{\lambda}F_{\lambda,k}(t) - 2\left(|b(t)| + \frac{|c'_{1/n_k}(t)|}{c_{1/n_k}(t)}\right)F_{\lambda,k}(t) \\
&\geq -\left(\frac{2\sqrt{\lambda}}{c_0}(\sup_{t \geq 0} c(t))|c_{1/n_k}(t) - c(t)| - 2|b(t)| - 2\frac{|c'_{1/n_k}(t)|}{c_0}\right)F_{\lambda,k}(t),
\end{aligned}$$

for almost every $t \in (0, \infty)$, where $c_0 = \inf_{t \geq 0} c(t) (> 0)$. Hence, observing (3.2), (3.3) and the absolute continuity of $F_{\lambda,k}(t)$ with respect to t , we obtain

$$\begin{aligned}
F_{\lambda,k}(t) &\geq F_{\lambda,k}(S) \exp\left(-\frac{2\sqrt{\lambda}}{c_0 n_k} \sup_{t \geq 0} c(t) \text{Var}(c; [0, \infty))\right. \\
&\quad \left.- 2\|b\|_{L^1(0, \infty)} - \frac{2}{c_0} \text{Var}(c; [0, \infty))\right)
\end{aligned}$$

for every $t \geq S$. Letting $k \rightarrow \infty$ in the inequality above, and observing (3.4), we obtain

$$F_\lambda(t) \geq F_\lambda(S) \exp\left(-2\|b\|_{L^1(0, \infty)} - \frac{2}{c_0} \text{Var}(c; [0, \infty))\right),$$

for every $t \geq S$ satisfying $t \notin N_1$. Letting $\lambda \rightarrow \infty$ in the above inequality yields

$$F(t) \geq F(S) \exp\left(-2\|b\|_{L^1(0, \infty)} - \frac{2}{c_0} \text{Var}(c; [0, \infty))\right),$$

for every $t \geq S$ satisfying $t \notin N_1$, which together with (3.23) implies that

$$(y_{1, \infty}, y_{2, \infty}) \neq (0, 0). \quad (3.24)$$

We next prove

$$c_* = c_\infty. \quad (3.25)$$

By the expression (3.15), we have

$$\|\mathcal{A}^{1/2}v(t)\|^2 = \|\phi\|^2 + \|\psi\|^2 + 2\text{Re}(e^{2ic_*tA^{1/2}}\phi, \psi)_H \quad (3.26)$$

$$\|v'(t)\|^2 = c_*^2\left(\|\phi\|^2 + \|\psi\|^2 - 2\text{Re}(e^{2ic_*tA^{1/2}}\phi, \psi)_H\right). \quad (3.27)$$

By Lemma 2 with $g(t) \equiv 2c_*$, we have

$$\begin{aligned} \left| \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (e^{2ic_*tA^{1/2}} \phi, \psi)_H dt \right| &= \left| \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{2ic_*tA^{1/2}} \phi dt, \psi \right)_H \right| \\ &\leq \left\| \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{2ic_*tA^{1/2}} \phi dt \right\| \|\psi\| = 0. \end{aligned} \quad (3.28)$$

Thus,

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T (e^{2ic_*tA^{1/2}} \phi, \psi)_H dt = 0,$$

which together with (3.26) and (3.27) yields

$$c_*^2 \lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T \|\mathcal{A}^{1/2} v(t)\|^2 dt = \lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T \|v'(t)\|^2 dt. \quad (3.29)$$

Put

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} := Y(t) \mathbf{y}_\infty = \begin{pmatrix} e^{i\tau(t)A^{1/2}} y_{1,\infty} + e^{-i\tau(t)A^{1/2}} y_{2,\infty} \\ ic(t) e^{i\tau(t)A^{1/2}} y_{1,\infty} - ic(t) e^{-i\tau(t)A^{1/2}} y_{2,\infty} \end{pmatrix}.$$

Then

$$\begin{aligned} \|w_1(t)\|^2 &= \|y_{1,\infty}\|^2 + \|y_{2,\infty}\|^2 + 2 \operatorname{Re}(e^{2i\tau(t)A^{1/2}} y_{1,\infty}, y_{2,\infty})_H, \\ \|w_2(t)\|^2 &= c(t)^2 \left(\|y_{1,\infty}\|^2 + \|y_{2,\infty}\|^2 - 2 \operatorname{Re}(e^{2i\tau(t)A^{1/2}} y_{1,\infty}, y_{2,\infty})_H \right). \end{aligned} \quad (3.30)$$

Using Lemma 2 with $g(t) = 2c(t)$, we have in the same way as in (3.28),

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T (e^{2i\tau(t)A^{1/2}} y_{1,\infty}, y_{2,\infty})_H dt = 0.$$

Thus (3.30), (3.31), (3.24) and the convergence $c_\infty = \lim_{t \rightarrow \infty} c(t)$ yield

$$\begin{aligned} c_\infty^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|w_1(t)\|^2 dt &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|w_2(t)\|^2 dt \\ &= c_\infty^2 \left(\|y_{1,\infty}\|^2 + \|y_{2,\infty}\|^2 \right) \neq 0. \end{aligned} \quad (3.32)$$

From the expression (3.13) with (3.12) and the boundedness of the operator $Y(t)$ uniformly to $t \geq 0$, it follows that

$$\lim_{t \rightarrow \infty} \|\mathcal{A}^{1/2} u(t) - w_1(t)\| + \lim_{t \rightarrow \infty} \|u'(t) - w_2(t)\| = 0,$$

which together with (3.32) yields

$$c_\infty^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\mathcal{A}^{1/2} u(t)\|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|u'(t)\|^2 dt \neq 0.$$

The equality above and (2.6) imply

$$c_\infty^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\mathcal{A}^{1/2} v(t)\|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|v'(t)\|^2 dt \neq 0. \quad (3.33)$$

Comparing (3.29) and (3.33), we obtain (3.25).

Now we prove (1.9) under the assumption

$$y_{1,\infty} \neq 0. \quad (3.34)$$

The case $y_{1,\infty} = 0$ and $y_{2,\infty} \neq 0$ can be treated in the same way. Put

$$f(t) := \int_0^t (c(s) - c_\infty) ds = \tau(t) - c_\infty t \quad \text{for } t \geq 0.$$

Then $f \in C([0, \infty))$. We put

$$\alpha = \liminf_{t \rightarrow \infty} f(t), \quad \beta = \limsup_{t \rightarrow \infty} f(t) \quad (\in [-\infty, \infty]).$$

It suffices to show

$$\alpha = \beta \in R. \quad (3.35)$$

First we show that $\beta < \infty$. Suppose that $\beta = \infty$. Since f is continuous and Lebesgue measure of N_1 is zero, we can take sequences $\{t_k\}_{k \in \mathbb{N}}$ such that

$$t_k \notin N_1, \quad \lim_{k \rightarrow \infty} t_k = \infty, \quad \lim_{k \rightarrow \infty} f(t_k) = \infty.$$

Let γ be an arbitrary positive number. For every $k \in \mathbb{N}$, since $\lim_{n \rightarrow \infty} f(t_k + n) = \infty$, the intermediate value theorem implies that there is $s_k > t_k$ satisfying

$$f(s_k) = f(t_k) + \gamma.$$

By using the continuity of f at s_k and the fact that measure of N_1 is zero, we can take r_k such that

$$r_k \notin N_1, \quad r_k > t_k, \quad |f(r_k) - f(t_k) - \gamma| = |f(r_k) - f(s_k)| < \frac{1}{k}. \quad (3.36)$$

By (3.25), Lemma 3 yields (3.20). This implies

$$\lim_{t \notin N_1, t \rightarrow \infty} e^{-if(t)A^{1/2}} \phi = \lim_{t \notin N_1, t \rightarrow \infty} e^{-i(\tau(t) - c_\infty t)A^{1/2}} \phi = y_{1,\infty} \quad \text{in } H,$$

since $e^{itA^{1/2}}$ is a unitary operator on H . Hence, letting $k \rightarrow \infty$ in the equality

$$e^{i(f(r_k)-f(t_k)-\gamma)A^{1/2}} e^{-if(r_k)A^{1/2}} \phi = e^{-i\gamma A^{1/2}} e^{-if(t_k)A^{1/2}} \phi,$$

and observing (3.36) and the continuity of the unitary operator $e^{isA^{1/2}}$ with respect to s , we obtain

$$y_{1,\infty} = e^{-i\gamma A^{1/2}} y_{1,\infty}.$$

Thus, we have

$$(I + A^{1/2})^{-1} y_{1,\infty} = e^{-i\gamma A^{1/2}} (I + A^{1/2})^{-1} y_{1,\infty}.$$

Since $\gamma > 0$ is arbitrary, and since $(I + A^{1/2})^{-1} y_{1,\infty} \in D(A^{1/2})$, we differentiate the equality above with respect to γ to obtain

$$0 = \frac{d}{d\gamma} e^{-i\gamma A^{1/2}} (I + A^{1/2})^{-1} y_{1,\infty} = -iA^{1/2} e^{-i\gamma A^{1/2}} (I + A^{1/2})^{-1} y_{1,\infty}$$

on $(0, \infty)$. This implies that $y_{1,\infty} = 0$ by the injectivity of $A^{1/2}$ and $e^{-i\gamma A^{1/2}}$, which contradicts (3.34).

The assumption $\alpha = -\infty$ deduces contradiction in the same way.

We finally prove (3.35). The above facts imply that $\alpha, \beta \in \mathbb{R}$. Suppose that (3.35) fails to hold. Then the interval (α, β) is not empty. Let γ be an arbitrary number $\gamma \in (\alpha, \beta)$. For every $k \in \mathbb{N}$, the intermediate value theorem implies that there exists $s_k > k$ satisfying $f(s_k) = \gamma$. Then by the same reason as (3.36), we can take r_k such that

$$r_k \notin N_1 \quad r_k > k, \quad |f(r_k) - \gamma| = |f(r_k) - f(s_k)| < \frac{1}{k}. \quad (3.37)$$

Letting $k \rightarrow \infty$ in the equality

$$e^{-i(f(r_k)-\gamma)A^{1/2}} e^{i(\tau(r_k)-c_\infty r_k)A^{1/2}} y_{1,\infty} = e^{i\gamma A^{1/2}} y_{1,\infty},$$

and observing (3.20), (3.37) and the continuity of $e^{itA^{1/2}}$ with respect to t , we obtain

$$\phi = e^{i\gamma A^{1/2}} y_{1,\infty}.$$

Hence we have

$$(I + A^{1/2})^{-1} \phi = e^{i\gamma A^{1/2}} (I + A^{1/2})^{-1} y_{1,\infty}. \quad (3.38)$$

Since $\gamma \in (\alpha, \beta)$ is arbitrary and since $(I + A^{1/2})^{-1}H \subset D(A^{1/2})$, we differentiate (3.38) with respect to γ to obtain

$$iA^{1/2}e^{i\gamma A^{1/2}}(I + A^{1/2})^{-1}y_{1,\infty} = \frac{d}{d\gamma}e^{i\gamma A^{1/2}}(I + A^{1/2})^{-1}y_{1,\infty} = 0$$

on (α, β) . This implies that $y_{1,\infty} = 0$ by the injectivity of $A^{1/2}$ and $e^{i\gamma A^{1/2}}$, which contradicts to (3.34). \square

4 Appendix

In the case $b(t)$ is an integrable C^1 function and c is a C^1 function satisfying (1.1), it is clear that there exists a unique solution of initial value problem (2.4)–(2.5), equivalently, (2.1)–(2.2). Namely, the following proposition holds.

Proposition A. *Let $b(t)$ be an integrable C^1 function on $[0, \infty)$. Let $c(t)$ be a C^1 function satisfying (1.1). Then for every $(\phi_0, \psi_0) \in \mathcal{H}_{1/2} \times H$, the Cauchy problem (2.1)–(2.2) has a unique global weak solution. Furthermore, if $(A^{1/2}\phi_0, \psi_0) \in D(A^{J/2}) \times D(A^{J/2})$ for $J \geq 1$, the following holds.*

$$(A^{1/2}u, u') \in \bigcap_{j=0,1} C^j([0, \infty); \mathbb{H}_{(J-j)/2} \times \mathbb{H}_{(J-j)/2}).$$

On the existence of solutions of the Cauchy problem (1.2)–(1.3) under the assumption that $c(t)$ is of bounded variation, there are some results. Colombini, De Giorgi and Spagnolo [4] showed the existence of solution

$$\frac{\partial^2 u}{\partial t^2}(t, x) - \sum_{i,j=1}^n a_{i,j}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(t, x) \text{ in } [0, \infty) \times \mathbb{R}^n,$$

in the class $u \in C([0, T], H_{\text{loc}}^{s+1})$, $\partial u / \partial t \in L^2([0, T], H_{\text{loc}}^s)$ and $\partial^2 u / \partial t^2 \in L^1([0, T], H_{\text{loc}}^{s-1})$, where $a_{i,j}(t)$ is of bounded variation and

$$a_{i,j}(t) = a_{j,i}(t), \quad \sum_{i,j=1}^n a_{i,j}(t) \xi_i \xi_j \geq \lambda_0 |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n,$$

for $\lambda_0 > 0$. In the case A is a corecive self-adjoint operator, De Simon and Torelli [5] showed the unique existence of the solution of (1.2)–(1.3) in the class $u \in W^{1,2}([0, T], H)$, $\partial u / \partial t \in L^2([0, T], D(A^{1/2}))$. Arosio [1]

considered (1.6)–(1.8) with $(\phi_0, \psi_0) \in H_0^1(\Omega) \times L^2(\Omega)$ for bounded domain Ω , and showed the unique existence of solution in the class $u \in C([0, \infty), H_0^1(\Omega)) \cap C^1([0, \infty), L^2(\Omega))$. The results above ([4], [5] and [1]) considered the solutions in the sense of distribution with respect to t . On the other hand, Bárta [2, section 2] considered the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{i,j}(t, x) \frac{\partial u}{\partial x_j} \right) (t, x) + \sum_{i=1}^n p_i(t, x) \frac{\partial u}{\partial x_i}(t, x) + q(t, x)u(t, x) \quad \text{in } (0, T) \times \Omega, \quad (4.1)$$

$$u(0, x) = \phi_0(x), \quad \frac{\partial u}{\partial t}(0, x) = \psi_0(x) \quad \text{in } \Omega, \quad (4.2)$$

where Ω is a bounded domain in \mathbb{R}^n , and $a_{i,j}$, p_i and q are functions satisfying the following:

$$\begin{aligned} a_{i,j} &\in BV([0, \infty), W^{1,\infty}) \cap L^\infty([0, \infty), Lip(\Omega)), \\ a_{i,j}(t, x) &= a_{j,i}(t, x), \quad \sum_{i,j=1}^n a_{i,j}(t, x) \xi_i \xi_j \geq \lambda_0 |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \\ p_i, q &\in BV([0, \infty), L^\infty). \end{aligned}$$

Then he showed the unique existence of the solution $u(t) \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ of (4.1) with initial value in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, such that for an at most countable subset N ,

$$(u(t), u'(t)) \in C([0, \infty) \setminus N; (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)),$$

and $u'(t)$ is differentiable with values in $L^2(\Omega)$ at $t \in [0, \infty) \setminus N$. Bárta [2] proved this by showing and applying an abstract theorem.

Theorem B (Proposition 1.3 and Corollary 1.4 of [2]). *Let X, Y be uniformly convex Banach space. Let $\{\mathbb{A}(t)\}_{t \geq 0}$ be the family of closed operators in X with domain $D(\mathbb{A}(t)) \equiv Y$. Assume that the following conditions (i)–(iii) hold.*

- (i) *For every $t \geq 0$, $D(\mathbb{A}(t))$ is dense in X , and $\{\mathbb{A}(t)\}_{t \geq 0}$ is stable with constants $\beta, 1$, that is, the semi-infinite interval (β, ∞) belongs to the resolvent set of $-\mathbb{A}(t)$ and*

$$\|(\mathbb{A}(t) + \xi)^{-1}\|_{\mathcal{L}(X)} \leq (\xi - \beta)^{-1}, \quad \xi > \beta,$$

for every $t \geq 0$.

- (ii) *There exists a family of uniformly convex Banach spaces $X_t = (X, \|\cdot\|_t)$ and a function of bounded variation $a : [0, \infty) \rightarrow \mathbb{R}$ such that*

$$\frac{\|x\|_t}{\|x\|_s} \leq e^{|a(t)-a(s)|}$$

holds for all $x \in X$ and $0 \leq s, t \leq T$.

- (iii) *The mapping $t \mapsto \mathbb{A}(t)$ is of bounded variation with values in $B(Y, X)$.*

Then there exists a family operators $U(t, s) \in B(X)$, $(t, s) \in \Delta = \{(t, s) \in \mathbb{R}^2; 0 \leq s \leq t \leq T\}$ such that the following (a)–(c) hold.

- (a) *$U(t, s)$ is strongly continuous in X with respect to s, t , $U(t, t) = I$ and $\|U(t, s)\|_X \leq e^{\beta(t-s)}$.*
- (b) *$U(t, s)Y \subset Y$ and $\|U(t, s)\|_Y \leq e^{\beta(t-s)}$.*
- (c) *For every $y \in Y$, there exists a countable set $N_y \subset [0, \infty)$ such that the mapping $t \mapsto U(t, s)y$ is continuous in the norm of Y , and that $D_t U(t, s)y = -\mathbb{A}(t)U(t, s)y$ holds for all $(t, s) \in \Delta$, $t \notin N_y$.*

As is stated above, Bárta [2] applied Theorem B to the hyperbolic equation (4.1) to show the unique existence of solutions. Similarly, we can apply Theorem B to the Cauchy problem (2.1)–(2.2) to obtain the solution $u(t)$. In the argument of this paper, we need the fact that $u'(t)$ is absolutely continuous with value in $\mathbb{H}_{-1/2}$. This fact is verified by the following lemma, which is proved at the end.

Lemma 4. *Let X be a separable Banach space. Assume that $f(t)$ is an X -valued continuous function on $[a, b]$ and that $g(t)$ is an X -valued integrable function on (a, b) . Assume moreover that there exists an at most countable subset N of $[a, b]$ such that $g(t)$ is continuous on $[a, b] \setminus N$ and that $f(t)$ is differentiable on $(a, b) \setminus N$ with $f'(t) = g(t)$. Then $f(t)$ is absolutely continuous on $[a, b]$, and satisfies*

$$f(t) = \int_a^t g(s)ds + f(a) \quad \text{for } t \in [a, b].$$

Now we state a proposition on the unique existence of the solution (2.1)–(2.2).

Proposition 5. *Let $b(t)$ be of bounded variation and integrable on $[0, \infty)$. Let $c(t)$ be of bounded variation on $[0, \infty)$ satisfying (1.1). Then the following assertions hold. For every $(\phi_0, \psi_0) \in \mathcal{H}_{1/2} \times D(A^{1/2})$ satisfying $\mathcal{A}^{1/2}\phi_0 \in D(A^{1/2})$, the Cauchy problem (2.1)–(2.2) has a unique global weak solution. Furthermore, $u' \in AC_{\text{loc}}([0, \infty); H)$ and there exists an at most countable subset N such that*

$$(\mathcal{A}^{1/2}u(t), u'(t)) \in C([0, \infty) \setminus N; \mathbb{H}_{1/2} \times \mathbb{H}_{1/2}),$$

and (2.1) holds in the space H at every $t \in [0, \infty) \setminus N$.

Proof. Let

$$X_t \equiv X = H \times H, \quad Y = \mathbb{H}_{1/2} \times \mathbb{H}_{1/2},$$

with inner product on X_t

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)_t := c(t)^2(x_1, y_1)_H + (x_2, y_2)_H.$$

We define

$$\mathbb{A}(t) = \begin{pmatrix} 0 & -A^{1/2} \\ c(t)^2 A^{1/2} & b(t) \end{pmatrix} \quad \text{with domain } D(\mathbb{A}(t)) = Y.$$

Then in the same way as in the proof of [2, section2], we see that the assumption of Theorem B are satisfied. Let $\{U(t, s) \in B(H \times H); 0 \leq s \leq t\}$ be a family of evolution operators given by Theorem B. Put

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix} := U(t, 0) \begin{pmatrix} \mathcal{A}^{1/2}\phi_0 \\ \psi_0 \end{pmatrix}.$$

Then

$$\mathbf{x}(0) = \begin{pmatrix} \mathcal{A}^{1/2}\phi_0 \\ \psi_0 \end{pmatrix} \in Y,$$

and thus, Theorem B implies $\mathbf{x}(t) \in Y$ with

$$\|\mathbf{x}(t)\|_Y \leq \|U(t, 0)\|_{\mathcal{L}(Y)} \left\| \begin{pmatrix} \mathcal{A}^{1/2}\phi_0 \\ \psi_0 \end{pmatrix} \right\|_Y \leq e^{\beta t} \left\| \begin{pmatrix} \mathcal{A}^{1/2}\phi_0 \\ \psi_0 \end{pmatrix} \right\|_Y \quad (4.3)$$

for every $t \geq 0$, and there exists at most countable set N_0 depending on initial data such that

$$\mathbf{x}(t) \in C([0, \infty); H \times H) \cap C([0, \infty) \setminus N_0; \mathbb{H}_{1/2} \times \mathbb{H}_{1/2}), \quad (4.4)$$

and that $\mathbf{x}(t)$ is differentiable on $[0, \infty) \setminus N_0$ and satisfies

$$\frac{d}{dt}\mathbf{x}(t) + \mathbb{A}(t)\mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } H \times H, \quad t \in [0, \infty) \setminus N_0.$$

Since c is of bounded variation on $[0, \infty)$, there is an at most countable set $N_c \subset [0, \infty)$ such that $c \in C([0, \infty) \setminus N_c)$. Thus, by (4.3) and (4.4), we see that

$$\mathbb{A}(t)\mathbf{x}(t) \in C([0, \infty) \setminus (N_0 \cup N_c); H \times H) \cap L_{\text{loc}}^1([0, \infty); H \times H).$$

Hence,

$$\frac{d}{dt}\mathbf{x}(t) = -\mathbb{A}(t)\mathbf{x}(t) \in C([0, \infty) \setminus (N_0 \cup N_c); H \times H) \cap L_{\text{loc}}^1([0, \infty); H \times H). \quad (4.5)$$

This fact and (4.4) with the aid of Lemma 4 imply $\mathbf{x}(t) \in AC_{\text{loc}}([0, \infty); H \times H)$. We define

$$u(t) := \int_0^t v(s)ds + \phi_0. \quad (4.6)$$

Then

$$u'(t) = v(t) \in AC_{\text{loc}}([0, \infty); H).$$

Since $v(t)$ is bounded in $\mathbb{H}_{1/2}$ by (4.3), we see that $u(t) - \phi_0 \in AC_{\text{loc}}([0, \infty); \mathbb{H}_{1/2})$. Since w is absolutely continuous,

$$\mathcal{A}^{1/2}u(t) = \int_0^t \mathcal{A}^{1/2}v(s)ds + \mathcal{A}^{1/2}\phi_0 = \int_0^t w'(s)ds + \mathcal{A}^{1/2}\phi_0 = w(t), \quad (4.7)$$

for every $t \in [0, \infty)$. From (4.5)–(4.7), it follows that u satisfies (2.1) in H for all $t \in (0, \infty) \setminus (N_0 \cup N_c)$. From the argument above, we see that u is a weak solution of (2.1)–(2.2) and belongs to the class stated in Proposition 5.

The uniqueness of the solution is easily seen by Gronwall's inequality. \square

Proof of Lemma 4. Fix an arbitrary positive number ε . Since $g(t)$ is integrable, there exists a positive number γ such that the estimate

$$\int_E \|g(t)\| dx < \frac{\varepsilon}{4} \text{ if } \mu(E) < \gamma \quad (4.8)$$

holds, where $\mu(E)$ denotes the Lebesgue measure of E for Lebesgue measurable set $E \subset \mathbb{R}$. Put $a = t_1$, $b = t_2$ and $N = \{t_j \mid j = 3, 4, \dots\}$. Since $f(t)$

is uniformly continuous on $[a, b]$, there exists a positive number $\delta_j < \gamma/2^{j+1}$ for every $j = 1, 2, \dots$ such that the estimate $\|f(t) - f(s)\| < \varepsilon/2^{j+2}$ holds for every $s, t \in [a, b]$ satisfying $|t - s| < 2\delta_j$. On the other hand, for every

$$c \in S := (a, b) \setminus \bigcup_{j=1}^{\infty} (t_j - \delta_j, t_j + \delta_j),$$

the function $f(t)$ is differentiable at $t = c$, and $f'(t) = g(t)$ is continuous at $t = c$. Hence there exists a positive number $\delta(c)$ such that the inequalities

$$\|f(t) - f(c) - (t - c)g(c)\| \leq \frac{\varepsilon|t - c|}{8(b - a)}, \quad \|g(t) - g(c)\| \leq \frac{\varepsilon}{8(b - a)} \quad (4.9)$$

hold for every $t \in (c - \delta(c), c + \delta(c)) \cap [a, b]$. Then we have

$$\bigcup_{j=1}^{\infty} (t_j - \delta_j, t_j + \delta_j) \cup \bigcup_{c \in S} (c - \delta(c), c + \delta(c)) \supset [a, b].$$

Hence we can choose a finite subset J_0 of \mathbb{N} and a finite sequence $\{c_k \in S\}_{k=1}^M$ satisfying $a < c_1 < c_2 < \dots < c_M < b$ such that

$$\bigcup_{j \in J_0} (t_j - \delta_j, t_j + \delta_j) \cup \bigcup_{k=1}^M (c_k - \delta(c_k), c_k + \delta(c_k)) \supset [a, b]. \quad (4.10)$$

Let (J, K) be a minimal pair of set such that $J \subset J_0$, $K \subset \{1, 2, \dots, M\}$ satisfying

$$\bigcup_{j \in J} (t_j - \delta_j, t_j + \delta_j) \cup \bigcup_{k \in K} (c_k - \delta(c_k), c_k + \delta(c_k)) \supset [a, b].$$

Put

$$\begin{aligned} \mathcal{I} &= \{(\alpha_m, \beta_m) \mid m = 1, \dots, L\} \\ &= \{(t_j - \delta_j, t_j + \delta_j) \mid j \in J\} \cup \{(c_k - \delta(c_k), c_k + \delta(c_k)) \mid k \in K\}. \end{aligned}$$

Renumbering if necessary, we can assume that

$$\alpha_m < \alpha_{m+1}, \quad \beta_m < \beta_{m+1}$$

for $m = 1, 2, \dots, L - 1$. By the minimality, we see that $\alpha_1 < a \leq \alpha_2$, $\beta_{L-1} \leq b < \beta_L$. We also have

$$\alpha_m < \beta_{m-1} \leq \alpha_{m+1} < \beta_m$$

for every $m = 2, \dots, L-1$. In fact, if $\alpha_{m+1} < \beta_{m-1}$, then we have

$$\alpha_{m-1} < \alpha_m < \alpha_{m+1} < \beta_{m-1} < \beta_m < \beta_{m+1}.$$

It follows that $(\alpha_m, \beta_m) \subset (\alpha_{m-1}, \beta_{m-1}) \cup (\alpha_{m+1}, \beta_{m+1})$, which contradicts the minimality of \mathcal{I} .

We now choose a sequence $\{p_m\}_{m=0}^L$ satisfying $a = p_0 < p_1 < \dots < p_L = b$ such that $\alpha_{m+1} < p_m < \beta_m$ holds for every $m = 1, \dots, L-1$. Here we note

$$\alpha_m < p_{m-1} < p_m < \beta_m \quad (4.11)$$

for every $m = 1, \dots, L$. Furthermore, we can choose $\{p_m\}_{m=1}^{L-1}$ so that $p_{m-1} \leq c_k \leq p_m$ holds if (α_m, β_m) is of the form $(c_k - \delta(c_k), c_k + \delta(c_k))$. Put

$$\begin{aligned} \Lambda &:= \{m \mid (\alpha_m, \beta_m) = (t_{j(m)} - \delta_{j(m)}, t_{j(m)} + \delta_{j(m)}) \text{ with some } j(m)\}, \\ P &:= \{m \mid (\alpha_m, \beta_m) = (c_{k(m)} - \delta(c_{k(m)}), c_{k(m)} + \delta(c_{k(m)})) \\ &\quad \text{with some } k(m) \in K\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\left\| f(b) - f(a) - \int_a^b g(s) ds \right\| \\ &= \left\| \sum_{m=1}^L \{f(p_m) - f(p_{m-1})\} - \int_a^b g(s) ds \right\| \leq I_1 + I_2 + I_3, \quad (4.12) \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{m \in \Lambda} \|f(p_m) - f(p_{m-1})\|, \\ I_2 &= \left\| \int_E g(s) ds \right\| \quad \text{with } E = \bigcup_{m \in \Lambda} [p_{m-1}, p_m], \\ I_3 &= \sum_{m \in P} \left\| f(p_m) - f(p_{m-1}) - \int_{p_{m-1}}^{p_m} g(s) ds \right\|. \end{aligned}$$

Observing (4.11), we have

$$I_1 < \sum_{j=1}^{\infty} \sup_{t, s \in (t_j - \delta_j, t_j + \delta_j)} \|f(t) - f(s)\| < \frac{\varepsilon}{4}. \quad (4.13)$$

Next, since

$$\mu(E) = \sum_{m \in \Lambda} (p_m - p_{m-1}) < \sum_{m \in \Lambda} (\beta_m - \alpha_m) < \sum_{j=1}^{\infty} 2\delta_j < \gamma,$$

inequality (4.8) implies

$$I_2 = \left\| \int_E g(s) ds \right\| < \frac{\varepsilon}{4}. \quad (4.14)$$

Finally, we treat the case that $m \in P$, that is, $(\alpha_m, \beta_m) = (c_k - \delta(c_k), c_k + \delta(c_k))$ holds with some $k = k(m) \in K$. In this case we have

$$c_k - \delta(c_k) < p_{m-1} \leq c_k \leq p_m < c_k + \delta(c_k).$$

Then observing (4.9), we have

$$\begin{aligned} & \left\| f(p_m) - f(c_k) - \int_{c_k}^{p_m} g(s) ds \right\| \\ & \leq \|f(p_m) - f(c_k) - (p_m - c_k)g(c_k)\| + \int_{c_k}^{p_m} \|g(s) - g(c_k)\| ds \\ & \leq \frac{\varepsilon(p_m - c_k)}{4|b - a|}. \end{aligned}$$

In the same way we have

$$\left\| f(c_k) - f(p_{m-1}) - \int_{p_{m-1}}^{c_k} g(s) ds \right\| \leq \frac{\varepsilon(c_k - p_{m-1})}{4|b - a|}.$$

Summing up we obtain

$$\left\| f(p_m) - f(p_{m-1}) - \int_{p_{m-1}}^{p_m} g(s) ds \right\| \leq \frac{\varepsilon(p_m - p_{m-1})}{4|b - a|}$$

for every $m \in P$, which implies

$$I_3 \leq \frac{\varepsilon}{4(b - a)} \sum_{m \in P} (p_m - p_{m-1}) \leq \frac{\varepsilon}{4}. \quad (4.15)$$

Substituting (4.13), (4.14) and (4.15) into (4.12), we conclude

$$\left\| f(b) - f(a) - \int_a^b g(s) ds \right\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$f(b) - f(a) = \int_a^b g(s) ds.$$

Applying the same argument on $[a, t]$ for every $t \in [a, b]$, we obtain the conclusion. \square

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